PHYSICAL REVIEW E **72**, 025604(R) (2005)

Quintic complex Ginzburg-Landau model for ring fiber lasers

A. Komarov,* H. Leblond, and F. Sanchez
Laboratoire POMA, UMR 6136, Université d'Angers, 2 Boulevard Lavoisier, 49000 Angers, France
(Received 11 April 2005; published 26 August 2005)

The dynamics of a fiber ring laser mode locked by nonlinear polarization rotation is reduced to a quintic complex Ginzburg-Landau (CGLQ) equation. The coefficients of the equation are explicitly given as functions of the physical parameters of the laser, especially the orientation of the phase plates. Then known results about analytic solutions, stability of pulselike solutions, and bound states of the CGLQ equation are examined from the point of view of their dependence with regard to the physical parameters.

DOI: 10.1103/PhysRevE.72.025604 PACS number(s): 42.65.Tg, 42.55.Wd, 05.45.Yv

The nonlinear Schrödinger (NLS) equation is known as a universal model accounting for the propagation of pulses, in nonlinear, dispersive, and conservative media, within the slowly varying envelope approximation. When the medium presents linear and nonlinear gains or losses, the NLS equation becomes the so-called cubic complex Ginzburg-Landau (CGLC) one, which is thus a universal model describing the evolution of the envelope of a pulse in a nonconservative medium [1]. The CGLC equation has been studied for many years. The early studies concerned the modulational instability [2], leading to chaos [3], and to prove that the CGLC equation is not integrable in the sense of the soliton theory [4]. Despite its failure to pass the Painlevé test, Painlevé analysis has been used to study mathematical properties of the equation [5]. An analytical solution has been found: localized pulses or solitons, shock profiles, holes, or dark solitons [6,7]. A remarkable property is that, in contrast to the NLS equation, bright and dark solitons exist for both normal and anomalous dispersion. Still in contrast with the conservative NLS case, the formation of the soliton involves two equilibriums: between the Kerr effect and dispersion, but also between gains and losses. Therefore, the soliton has a fixed amplitude. A soliton with arbitrary amplitude has been found in the case where the linear excess of the gain is zero [8]. Other properties of the solitons, especially their stability, have been studied by treating the CGLC equation as a NLS one with nonconservative perturbations [9], and numerically [10,11].

The CGLC equation is an essential model for the study of laser dynamics, especially for fiber lasers, which are true one-dimensional media. A model very frequently used in this domain is the "master equation" derived by Haus *et al.* [12,13], which is nothing but the stationary version of the CGLC equation. The coefficients that appear in the model were related to the physical parameters in a rather phenomenological way [14]. A more rigorous derivation of the CGLC, in the case of a fiber laser mode locked by means of the nonlinear polarization rotation technique, has been given [15–17]. There the dependency of the coefficients of the CGLC with regard to the experimental parameters, especially the orientation of the phase plates, which allows us to control the polarization, has been explicitly determined. This allowed us to determine domains of relative stability of the solitons, which were in good agreement with the domain of mode locking observed experimentally [15]. It is only a relative stability: depending on the sign of the excess of the linear gain, the instability behaves in a very different manner. This gives a criterion that has been successfully compared with experimental results. However, the soliton solution of the CGLC is never stable, neither for anomalous dispersion [10], nor in the case of normal dispersion [11].

To achieve the stabilization of the solitons in the model, higher-order nonlinear (quintic) terms have been introduced. The obtained equation is referred to as the quintic (or cubic-quintic) complex Ginzburg-Landau (CGLQ) equation. It still possesses analytical solutions, which are generalizations of the fixed- and arbitrary-amplitude solitons of the CGLC [8], but also analytic solutions, flat top, algebraic, and chirp-free [10]. The stability of the solitons has been proved in the case of anomalous [18,19] and normal dispersion [11]. A lot of numerical studies also exist, among which we can cite the composite solitons [20]. An essential issue of the CGLQ model is the study of bound states of two solitons or more. Indeed, as soon as a bound state is not considered as a mere two-hump solution, but has the result of the interaction of two solitons behaving as quasiparticles, the stability of solitons is required. If not infinite, the lifetime of the quasiparticles must be large with regard to the duration of the interaction, which is very slow in this case. Bound states have been studied through a perturbative approach around the NLS equation [21], and by means of the energy-momentum balance approximation, for anomalous [22,23], and normal dispersion [24].

In any case, the coefficients of the quintic terms are given up to now in a purely phenomenological way. We published recently a model of the fiber ring laser mode locked through nonlinear polarization rotation, which allows us to account for multiple pulse operation, including the hysteresis and multistability [25]. This model gives an explicit dependency of the coefficients of the equations with regard to the physical parameters, especially the wave plates orientations. Using a weak amplitude approximation, this model can be reduced to the CGLQ equation.

Let us first recall its derivation as given in [25]. We consider an ytterbium-doped fiber ring laser passively mode

*Permanent address: Institute of Automation and Electrometry, Russian Academy of Sciences, Acad. Koptyug Pr., 1, 630090, Novosibirsk, Russia.

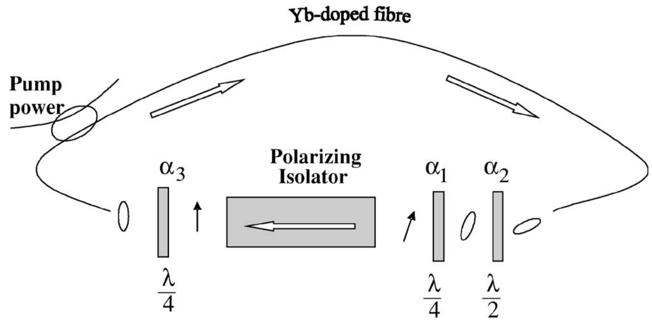

FIG. 1. Schematic representation of a fiber laser passively mode locked through nonlinear polarization rotation.

locked through a nonlinear polarization rotation. The setup is schematically shown in Fig. 1. After the polarizing isolator, the electric field has a linear polarization, which is transformed into an elliptical one by the quarter-wave plate 3. At the output of the fiber, the direction of the elliptical polarization of the central part of the pulse can be rotated towards the passing axis of the polarizer by the half-wave plate 2, then the ellipticity of the polarization is modified by the quarter-wave plate 1. We denote by α_1, α_2, and α_3 the angles between one eigenaxis of the plates 1, 2, and 3, respectively, and the passing axis of the polarizer. Assuming that nonlinearity dominates, the two components u and v of the electric field evolve in the fiber according to the equations

$$\frac{\partial u}{\partial z} = i\gamma[u|u|^2 + (1-B)u|v|^2 + Bv^2u^*], \quad (1)$$

$$\frac{\partial v}{\partial z} = i\gamma[v|v|^2 + (1-B)v|u|^2 + Bu^2v^*]. \quad (2)$$

γ ($W^{-1}\,m^{-1}$) is the nonlinear coefficient related to the nonlinear index coefficient n_2. In silicate fibers, $B=1/3$. Equations (1) and (2) can be solved analytically. The effects of the phase plates are treated by means of the Jones matrix formalism, and the electric field amplitude after the $(n+1)$th round trip $u=f_{n+1}$ is computed as a function of the same amplitude after the nth round trip f_n as

$$f_{n+1} = -\theta e^{i\gamma|f_n|^2 L}[\cos(\Omega L + \alpha)\cos(\alpha_1 - \alpha_3) + i\sin(\Omega L + \alpha)\sin(\alpha_1 + \alpha_3)]f_n, \quad (3)$$

where $\Omega = \gamma B|f_n|^2 \sin 2\alpha_3$, $\alpha = 2\alpha_2 - \alpha_1 - \alpha_3$. θ is the transmission coefficient of the polarizer, and L the length of the cavity. We assumed that the dispersion and gain have a small effect on a single round trip in the cavity, which allows us to approximate them as uniformly distributed along the cavity. This yields the following equation for the evolution of the amplitude f along the fiber:

$$\frac{\partial f}{\partial z} = \mathcal{L}f = \left(\frac{g_0}{\omega_g^2} - i\frac{\beta_2}{2}\right)\frac{\partial^2 f}{\partial t^2} + g_0 f, \quad (4)$$

where β_2 ($ps^2\,m^{-1}$) is the second-order group velocity dispersion (GVD), g_0 (m^{-1}) the linear gain. We neglect gain saturation. It depends indeed on the averaged power in the cavity. If the pulse length is short enough with regard to the cavity length, the averaged power is small, so that the approximation is justified. For each round trip, the differential equation (4) was solved numerically, then the algebraic transformation (3) applied to the result, and so on. The full detail of the above derivation is given in [25].

Here we modify the approach as follows: The discrete sequence $f_n(t)$ defined by relation (3) is interpolated by a continuous function $f(z,t)$. If relation (4) is omitted, the interpolation yields the differential equation

$$\frac{\partial f}{\partial z} = \mathcal{N}f = \left[i\gamma|f|^2 + \frac{1}{L}\ln|\theta\cos 2\alpha_3\cos(\Omega L + \alpha)|\right]f. \quad (5)$$

Taking into account the dispersion and gain terms of Eq. (4) as perturbations of the first approximation defined by Eq. (5), the linear evolution operator $\mathcal{L}$ and the nonlinear one $\mathcal{N}$ add up. We get the partial differential equation (PDE) to be satisfied by f, as

$$\frac{\partial f}{\partial z} = (\mathcal{L}+\mathcal{N})f = \left(\frac{g_0}{\omega_g^2} - i\frac{\beta_2}{2}\right)\frac{\partial^2 f}{\partial t^2} + \left[g_0 - \sigma + \frac{1}{L}\ln|\cos(p|f|^2 + \alpha)| + i\gamma|f|^2\right]f, \quad (6)$$

where we have set $\sigma = -\ln|\theta\cos 2\alpha_3|/L$, and $p = \gamma BL\sin 2\alpha_3$. Then we consider a weak amplitude approximation. The nonlinear term in Eq. (6) is expanded in a power series of $|f|^2$, and we get

$$\frac{\partial f}{\partial z} = \left(\frac{g_0}{\omega_g^2} - i\frac{\beta_2}{2}\right)\frac{\partial^2 f}{\partial t^2} + \left[g_0 - \sigma - \frac{\ln|\cos\alpha|}{L} - \frac{p\tan\alpha}{L}|f|^2 - \frac{p^2}{2L\cos^2\alpha}|f|^4 + i\gamma|f|^2\right]f, \quad (7)$$

Equation (7) is the CGLQ equation. It can be reduced to the dimensionless form used in [10,11], which is

$$i\frac{\partial\psi}{\partial\zeta} + \frac{D}{2}\frac{\partial^2\psi}{\partial\tau^2} + \psi|\psi|^2 = i\delta\psi + i\beta\frac{\partial^2\psi}{\partial\tau^2} + i\varepsilon\psi|\psi|^2 + i\mu\psi|\psi|^4, \quad (8)$$

by means of the following relations:

$$\psi = \sqrt{\gamma L}f, \quad \zeta = \frac{z}{L}, \quad \tau = \frac{t}{\sqrt{|\beta_2|L}}, \quad (9)$$

$$D = -\operatorname{sgn}\beta_2, \quad \beta = \frac{g_0}{\omega_g^2|\beta_2|}, \quad (10)$$

$$\delta = Lg_0 + \ln|\theta\cos 2\alpha_3\cos\alpha|, \quad (11)$$

$$\varepsilon = -B\sin 2\alpha_2 \quad \tan\alpha, \quad \mu = -B^2\frac{\sin^2 2\alpha_3}{2\cos^2\alpha}. \quad (12)$$

An effective quintic absorption-gain term has been derived this way, with the nonvanishing coefficient μ, without any quintic nonlinearity of the material. Thus the CGLQ equation can be valid even if the quintic nonlinearity of the medium is completely negligible. Notice that the effective

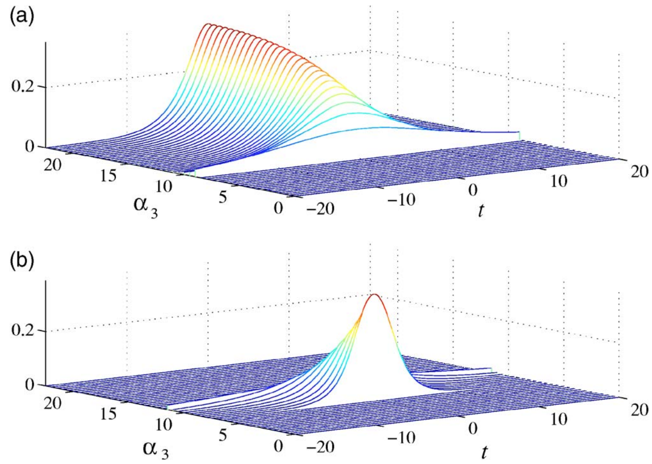

FIG. 2. (Color online) Time profile of the analytical solution of CGLQ against the angle α_3. Parameters: $\theta=0.95$, $L=9$ m, $\omega_g=10\ \mathrm{ps}^{-1}$, $\beta_2=0.026\ \mathrm{ps}^2\ \mathrm{m}^{-1}$, $g_0=4/9\ \mathrm{m}^{-1}$; $(\sigma,\eta)=(1,1)$ (a), $(-1,1)$ (b).

quintic nonlinear index, denoted by ν in [10,11], is zero in this approximation. The parameter set of the CGLQ is wide, this condition restricts it considerably, facilitating the analysis of the considered physical problem. The coefficient β depends on the gain filtering ρ and on the dispersion β_2, it can be adjusted experimentally to some extent. The coefficient ε can be adjusted to any value by a choice of the combination $\alpha=2\alpha_2-\alpha_1-\alpha_3$ of the angles of the phase plates. The excess of linear gain δ can be modified in two ways: acting on the pumping changes the gain coefficient g_0, while the losses can be modified by rotating one of the phase plates. If α is fixed, α_3 can be adjusted increasing arbitrarily

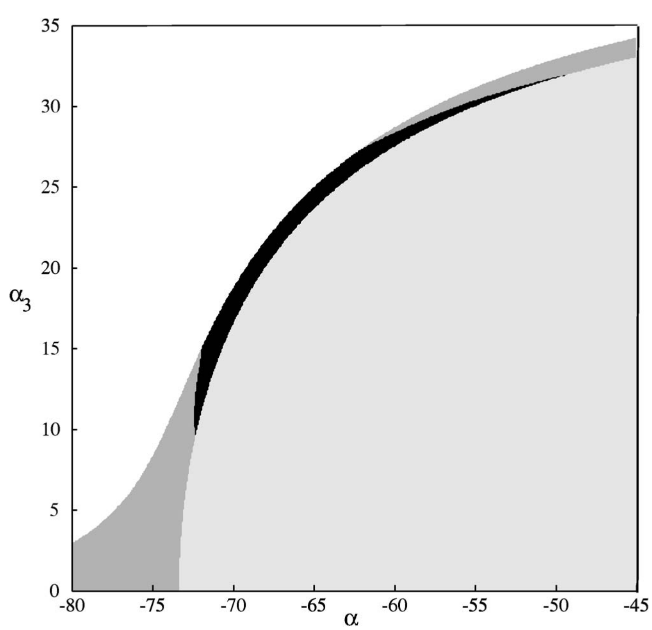

FIG. 3. Domains of existence of stable pulse solutions of CGLQ, against the angles. The parameters are $\theta=0.95$, $g_0L=1.3$, $L\beta_2=0.026\ \mathrm{ps}^2$, $\omega_g=10\ \mathrm{ps}^{-1}$. Black: stable pulses exist; gray: instability (in the light gray domain it is due to the background); white: stability is not determined.

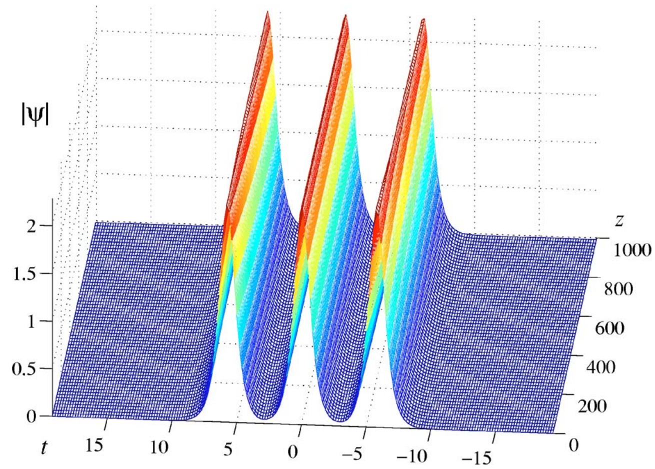

FIG. 4. (Color online) Propagation of a stable bound state of three pulses. The parameters are $\theta=0.95$, $g_0L=2.5$, $L\beta_2=-0.05\ \mathrm{ps}^2$, $\omega_g=10\ \mathrm{ps}^{-1}$, $(\alpha,\alpha_3)=(-85.047°,2.985°)$.

the losses. The value of the effective dimensionless quintic absorption-gain coefficient μ cannot be adjusted so easily. Indeed, we have the relation

$$\mu+\varepsilon^2+\frac{B^2}{2}\sin^2 2\alpha_3=0; \tag{13}$$

therefore μ can take values only between $-\varepsilon^2$ and $-\varepsilon^2-B^2/2$ (i.e., $-\varepsilon^2-1/18$ in silicate fibers). However, the CGLQ equation (8) is invariant under the transformation

$$\psi'=\frac{\psi}{a},\quad \zeta'=a^2\zeta,\quad \tau'=a\tau,\quad \delta'=\frac{\delta}{a^2},\quad \mu'=\mu a^2, \tag{14}$$

for any real value of a. Thus the parameter μ can be set to an arbitrary value by means of a change of the amplitude, time, and propagation distance scales, unless the sign of μ, the sign of δ, and the product $\mu\delta$ are fixed. Therefore any negative value of μ can be obtained. The parameters corresponding to a given set of values of ε, β and of the product $\delta\mu$ can be found as follows: first the expression (12) of ε gives α as an inverse tangent. Then Lg_0 is computed from the expressions (11) and (12) of δ and μ, and an expression of $L|\beta_2|$ vs α_3 is obtained from that of β, Eq. (10). The latter expression can be inverted numerically if $L|\beta_2|$ is given.

An analytical solution, known as the fixed-amplitude soliton, is given in [11] (its expression is too complicated to be reproduced here). It exists if

$$\varepsilon=-4\beta+3\sigma\sqrt{3+16\beta^2}, \tag{15}$$

with $\sigma=\pm1$. This can be achieved by choosing $\alpha=\arctan[\varepsilon/(B\sin 2\alpha_3)]$, where ε is a function of β, according to (15), and β depends on the linear gain g_0 through (10). There are four different expressions depending on the sign σ in (15), and on the choice of d_+ or d_- (see [11]). They do not yield a pulselike solution every time, especially, when a condition is required [Eq. (23) in [11]], which reduces here to $\delta\eta\sigma>0$, where η is the sign such that $d=d_\eta$. The analytical solutions are plotted against the orientation angle α_3 of the phase plate 3 in Fig. 2, in the case of normal dispersion. With the same values of the parameters, a flat-top solution is obtained for $\alpha_3 \simeq 7.1355$, and $(\sigma, \eta) = (-1, 1)$. It should take place at the front of the Fig. 2(b), but does not appear since the precision on α_3 required for its observation is very high.

However, the analytical solutions of the CGLQ equation are unstable. The existence of stable pulselike solutions has been considered numerically, especially in [11] for normal dispersion. The stability of pulses as a function of the angles α, α_3, i.e., the orientation of the phase plates, and of the linear gain g_0, can be deduced from such an analysis. Figure 3 presents an example of the conclusions that can be drawn from the results published in Ref. [11], without further numerical study of the CGLQ equation. The values of δ and μ considered appear to be quite restrictive, and imply a low dispersion, which can be obtained in the stretched pulse configuration. The white domain on Fig. 3 corresponds to values of $\delta\mu$ out of the range considered in [11], and we cannot conclude from these data only. The light gray domain corresponds to an unstable background ($\delta > 0$). The stability results of [11] can be applied to the central stripe. A domain of existence of stable pulses appears (in black).

Bound states have also been described. In [22,23], two sets of coefficients of the CGLQ equation are considered. For the same physical parameters as above, except a total dispersion $L\beta_2 = -0.05$ ps^2, anomalous, these coefficients can be obtained using $Lg_0 = 2.5$ and angles $(\alpha, \alpha_3) = (-83.957°, 17.432°)$ or $(-85.047°, 2.985°)$ respectively. Bound states of two and more pulses have been obtained for the former set, and it was shown that the two pulse states are unstable for the latter. However, in the latter case, triple bound states with a different phase symmetry can be found; a numerical example is shown on Fig. 4. The detail and the proof of the stability of this state are left for further publication.

Thus, considering a ring fiber laser mode locked through nonlinear polarization rotation, we have made the link between the physical parameters, which are mainly the orientation angles of the phase plates, and the coefficients of the CGLQ equation. This allows us to precisely understand more accurately in a concrete frame the meaning of the results of the mathematical physics about this equation, concerning analytic and numerical solutions and their stability, or bound states. The set of coefficients of the CGLQ equation is very wide, therefore it cannot be investigated systematically by numerical methods. The coefficients derived in this paper may give an orientation for further studies.

[1] A. C. Newell, Envelope Equations, Lect. Appl. Math. **15**, 157 (1974).
[2] J. T. Stuart, F. R. S. and R. C. DiPrima, Proc. R. Soc. London, Ser. A **362**, 27 (1978).
[3] K. Nozaki and N. Bekki, Phys. Rev. Lett. **51**, 2171 (1983).
[4] L. R. Keefe, Phys. Fluids **29**, 3125 (1986).
[5] R. Conte and M. Musette, Physica D **69**, 1 (1993).
[6] K. Nozaki and N. Bekki, J. Phys. Soc. Jpn. **53**, 1581 (1984).
[7] N. Bekki and K. Nozaki, Phys. Lett. **110**, 133 (1985).
[8] N. Akhmediev and V. V. Afanasjev, Phys. Rev. Lett. **75**, 2320 (1995).
[9] V. V. Afanasjev, Opt. Lett. **20**, 704 (1995).
[10] N. N. Akhmediev, V. V. Afanasjev, and J. M. Soto-Crespo, Phys. Rev. E **53**, 1190 (1996).
[11] J. M. Soto-Crespo, N. N. Akhmediev, V. V. Afanasjev, and S. Wabnitz, Phys. Rev. E **55**, 4783 (1997).
[12] H. A. Haus, J. G. Fujimoto, and E. P. Ippen J. Opt. Soc. Am. B **8**, 2068 (1991).
[13] H. A. Haus, J. G. Fujimoto, and E. P. Ippen, IEEE J. Quantum Electron. **28**, 2086 (1992).
[14] H. A. Haus, E. P. Ippen, and K. Tamura, IEEE J. Quantum Electron. **30**, 200 (1994).
[15] H. Leblond, M. Salhi, A. Hideur, T. Chartier, M. Brunel, and F. Sanchez, Phys. Rev. A **65**, 063811 (2002).
[16] M. Salhi, H. Leblond, and F. Sanchez, Phys. Rev. A **68**, 033815 (2003).
[17] M. Salhi, H. Leblond, F. Sanchez, M. Brunel, and A. Hideur, J. Opt. A, Pure Appl. Opt. **6**, 774 (2004).
[18] J. M. Soto-Crespo, N. N. Akhmediev, and V. V. Afanasjev, J. Opt. Soc. Am. B **13**, 1439 (1996).
[19] J. M. Soto-Crespo, N. Akhmediev, and G. Town, Opt. Commun. **199**, 283 (2001).
[20] N. Akhmediev, A. S. Rodrigues, and G. E. Town, Opt. Commun. **187**, 419 (2001).
[21] V. V. Afanasjev, B. A. Malomed, and P. L. Chu, Phys. Rev. E **56**, 6020 (1997).
[22] N. N. Akhmediev, A. Ankiewicz, and J. M. Soto-Crespo, Phys. Rev. Lett. **79**, 4047 (1997).
[23] N. N. Akhmediev, A. Ankiewicz, and J. M. Soto-Crespo, J. Opt. Soc. Am. B **15**, 515 (1998).
[24] A. Hideur, B. Ortaç, T. Chartier, M. Brunel, H. Leblond, and F. Sanchez, Opt. Commun. **225**, 71 (2003).
[25] A. Komarov, H. Leblond, and F. Sanchez, Phys. Rev. A **71**, 053809 (2005).